

Torons and D-brane bound states.

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We interpret instantons on a torus with twisted boundary conditions, in terms of bound states of branes. The interplay between the $SU(N)$ and $U(1)$ parts of the $U(N)$ theory of N 4-branes allows the construction of a variety of bound states. The $SU(N)$ and $U(1)$ parts can contribute fractional amounts to the total instanton number which is integral. The geometry of non-self intersecting two-cycles in T^4 sheds some light on a number of properties of these solutions.

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1. Introduction

Recent developments in string duality have shown that many aspects of gauge theories can be understood geometrically by using the fact that the low energy description of D-branes is Yang Mills theory [1][2]. Here we explore another aspect of this geometrization of Yang Mills by interpreting torons in terms of bound states of D-branes.

Torons are instantons on a torus [3], possibly with fractional instanton number, which exist when we impose twisted boundary conditions. They were studied by 't Hooft in $SU(N)/Z_N$ gauge theory. We discuss their embedding in $U(N)$ gauge theories, and interpret in terms of bound states of 4-branes, 2-branes and 0-branes. The fact that the global structure of the gauge group needed for the D-branes is $U(N)$, as opposed to $SU(N) \times U(1)$, will be crucial. In this paper our considerations will be mostly classical. We consider these solutions in the Euclidean T^4 of supersymmetric 4+1 dimensional Yang Mills (with 16 supersymmetries) on a torus $T^4 \times R$, and interpret in terms of 4-branes which have the four spatial dimensions along the T^4 . At this point we should comment on the interest in studying the classical solutions. The correct counting of quantum states in the sector specified by a set of brane charges is given by solving the quantum mechanics on the moduli space. This can often be reformulated in terms of the cohomology of an appropriate compactification of the moduli space. Superficially different but related cohomological formulations have been given in [4] and [5]. From this point of view, this paper is largely a step in investigating parts of these moduli spaces which can be realized simply in terms of fractional instantons. Placing fractional instantons in the D-brane context in this way leads to some simple geometric insights into the conditions of their existence.

The simplest class of bound states we consider will be denoted by (4220). These have 4-brane charge corresponding to 4-branes wrapped around the compact spatial directions 1, 2, 3, 4; 2-brane charge corresponding to the a 2-branes wrapped along the directions (12); 2-brane charge corresponding to 2-branes wrapped along the (34) direction; and zero brane charge. Such bound states have been discussed from the string theory point of view in [6] and from supergravity in [7]. In the context of the M(atrix) models of Banks, Fischler, Shenker and Susskind [8], systems with these charges have also been discussed recently [9][10][11].

The next class of solutions we consider has charges (422). These manage to have a vanishing zero brane charge thanks to a cancelation of the fractional instanton number between the $SU(N)$ and the $U(1)$ parts of the theory. They also manage to be supersymmetric configurations of the D-brane theory although they are not susy-configurations if we

take the susy transformations to be those of Yang Mills alone. This is possible because of the non-linearly realised supersymmetries, which have been emphasized recently in [5],[12].

Given the existence of the (422) system, T-duality leads us to expect the existence of (420) bound states. This leads us to look for supersymmetric solutions of $U(N)$ with magnetic flux in one 2-plane only. We indeed find that known solutions of $SU(N)/Z_N$ combined appropriately with fields in the $U(1)$ do indeed yield $U(N)$ solutions with the right properties.

T-duality also relates these systems carrying two-brane charges only. These relate the constraints on the existence of torons [3], to geometrical constraints on supersymmetric systems of intersecting branes at angles, which were first discussed in ref. [13].

While the instanton numbers in the $SU(N)/Z_N$ theory can be fractional, only combinations of $SU(N)$ and $U(1)$ fields which have integral total instanton number are solutions of the $U(N)$ gauge theory. This can be understood simply in terms of Dirac quantization of D-brane charges.

In section 5 we discuss the masses of these bound states, and their large N limit. We show, following arguments in [2], that the energy of the lowest lying states in certain topological sectors of the $SU(N)$ Yang Mills theory on $T^4 \times R$ vanishes in this limit.

2. Preliminary remarks

2.1. Brane charges and fluxes.

We will consider 4-branes aligned along the directions (1234), 2-branes aligned along (12), another 2-branes along (34), and 0-branes.

The low energy effective theory of N 4-branes is $U(N)$ Yang Mills, and we will look for the bound state by studying configurations in this theory. The presence of 2 branes along the (12) direction corresponds to a $U(1)$ magnetic flux in the (34) plane, and the presence of the 2 branes along the (12) plane corresponds to the presence of magnetic flux along the (34) plane. This follows from the Chern Simons couplings of the field strengths to the RR potentials [14],

$$\int C_{012} \text{tr} F_{34} + C_{034} \text{tr} F_{12}. \quad (2.1)$$

A field strength in the $U(1)$ acts as a source for the appropriate charge, but this field strength automatically implies that we have 't Hooft flux in the $SU(N)$ part, see [2],

[4], [15]. We will see this more explicitly below. Finally an instanton embedded in the directions (1234) acts as a source for the 0-brane. This follows from the interaction

$$\int C_0 \text{tr} F_{12} F_{34}. \quad (2.2)$$

The next step is to find solutions of the $U(N)$ theory corresponding to supersymmetric configurations of branes. This requires embedding 't Hooft's solutions of $SU(N)/Z_N$ into $U(N)$ gauge theory. As a preliminary to doing that we will discuss classical solutions of the $U(1)$ theory.

2.2. $U(1)$ solutions on a torus

Consider the configuration

$$\begin{aligned} A_1 &= B_{12}x_2 & A_2 &= 0 \\ A_3 &= B_{34}x_4 & A_4 &= 0, \end{aligned} \quad (2.3)$$

on a torus with sides of lengths a_1 , a_2 , a_3 , and a_4 . It satisfies non-trivial boundary conditions. Translation by a_2 in x_2 is accompanied by a gauge transformation by $e^{iB_{12}a_2x_1}$, and translation in x_4 is accompanied by a gauge transformation $e^{iB_{34}a_4x_3}$. Transporting a unit charge around the 12 plane gives the quantization condition $B_{12}a_1a_2 = 2\pi n_{12}$, whereas transporting in the (34) plane gives the condition $B_{34}a_3a_4 = 2\pi n_{34}$. Combining these two gives :

$$\frac{a_1a_2}{a_3a_4} = \frac{n_{12}}{n_{34}} \frac{B_{34}}{B_{12}} \quad (2.4)$$

If we further impose the self duality $B_{12} = B_{34}$, we find that the ratio of the box sizes is a rational number. However, self-duality of the $U(1)$ fields is not necessary for a supersymmetric configuration. As we will see in more detail later, requiring supersymmetry for $U(N)$ solutions will lead to similar constraints on box sizes. This follows by taking into account both the linear and nonlinear supersymmetries of the D-brane worldvolume theory, as given for example in [5]:

$$\delta\lambda = \xi_1 \Gamma^{MN} F_{MN} + \xi_2 1 \quad (2.5)$$

The second term is proportional to the unit matrix in the Lie algebra, so (2.5) allows constant field strengths in the $U(1)$, with no self-duality restriction, to give BPS configurations. If the field strengths live in the $SU(N)$ part of the algebra, a generic constant (in spacetime) field will not be susy, but self-dual or anti-self dual fields will yield susy configurations.

In the non-abelian case there can be solutions with $A_\mu^a = 0$ satisfying the twisted boundary conditions, as in first part of [3]. In the abelian case this is not possible, see [16]. This will also constrain the kind of supersymmetric solutions that can be constructed.

3. Embedding solutions in $U(N)$

In discussing the $U(N)$ theory we will recall that there is a map from $SU(N) \times U(1)$ to $U(N)$. Let $(U, e^{i\theta})$ be an element of the product group. Then the map from $SU(N) \times U(1)$ to $U(N)$ takes this to $e^{i\theta}U$ in $U(N)$. Notice that the elements $(e^{\frac{2i\pi n}{N}}, e^{-\frac{2i\pi n}{N}})$ are in the kernel of this map. Therefore we can arrange the twists to be trivial in $U(N)$ by cancelling them between $SU(N)$ and $U(1)$. This requires consistently combining solutions of $SU(N)/Z_N$ with $U(1)$ solutions similar to those described above, in such a way as to cancel the total twist.

One general remark about the instanton numbers of the solutions that we get in this way can be made immediately. Using the relations between the fields in the $U(1)$ and the twists, we have:

$$B_{\mu\nu}a_\mu a_\nu = \frac{2\pi n_{\mu\nu}}{N} \text{mod } (2\pi) \quad (3.1)$$

This allows us to express the contribution to the instanton number coming from the $U(1)$ fields in terms of the twists:

$$\frac{1}{16\pi^2} \int \text{tr} F_{\mu\nu} \tilde{F}_{\mu\nu} = \frac{1}{N} n_{\mu\nu} \tilde{n}_{\mu\nu} + \text{integer}, \quad (3.2)$$

where $F_{\mu\nu} = B_{\mu\nu}1$ is an $N \times N$ matrix. The sign is important. The fractional part coming from the $SU(N)$ is also determined in terms of the twists [3] to be $-\frac{1}{N} n_{\mu\nu} \tilde{n}_{\mu\nu}$. As a result the total instanton number is integral. Since the instanton number is related to zero brane charge inside a 4-brane, we may interpret the integrality of the total instanton number as a consequence of the existence of the six-brane of ten dimensional type IIA string theory with known charge, and Dirac quantization.

3.1. (4220) solutions with trivial $SU(N)$ gauge fields.

These bound states of 4, 2, and 0-branes have the property that the element of $H^2(T^4, Z)$ defined by the 2-brane charge has non-zero intersection number, but the total intersection number is zero. There are solutions of the $SU(N)/Z_N$ theory which have zero gauge fields in the presence of non-zero twists n_{12}, n_{34} where $\frac{n_{12}n_{34}}{N}$ is an integer. We briefly review the construction of these solutions [3]. Translation by a_μ in the x_μ direction is accompanied by a gauge transformation Ω_μ which are constrained by the equation :

$$\Omega_\mu(x_\nu = a_\nu)\Omega_\nu(x_\mu = 0) = \Omega_\nu(x_\mu = a_\mu)\Omega_\mu(x_\nu = 0)e^{\frac{i\pi n_{\mu\nu}}{N}}. \quad (3.3)$$

For vanishing gauge fields there are solutions of the form

$$\Omega_\mu = P^{s_\mu} Q^{t_\mu}. \quad (3.4)$$

where $PQ = QPe^{\frac{2i\pi}{N}}$. The equation (3.3) gives

$$n_{\mu\nu} = s_\mu t_\nu - s_\nu t_\mu \pmod{N} \quad (3.5)$$

A complete characterization of these solutions requires specifying the appropriate set of s and t parameters. A necessary condition for this equation to be soluble is

$$\frac{1}{8} \epsilon^{\mu\nu\alpha\beta} n_{\mu\nu} n_{\alpha\beta} = 0 \pmod{N}. \quad (3.6)$$

This means that the above $U(1)$ solutions can be simply embedded in the $U(N)$ theory, when the twists satisfy the above condition. With the twist chosen as above, the instanton number is $\frac{n_{12}n_{34}}{N}$. This gives a (4220) system. Since $SU(N)$ solution with non-trivial twists and zero gauge fields only exist when the product $n_{12}n_{34} = 0 \pmod{N}$ [3], requiring $U(N)$ gauge invariance implies that the zero brane charge can only be integral.

This system has the charges N 4-branes wrapped on (1234), n_{34} units of two-brane charge on the (12) cycle, n_{34} units of two-brane charge on the on (12) cycle, and $\frac{n_{12}n_{34}}{N}$ units of 0-brane charge. T-duality along the 1 and 3 directions gives a system with charges of two-branes in the (24), (14), (23) and (13) planes. We will discuss this relation in more detail in section 4.

3.2. (422) solutions with non-trivial $SU(N)$ gauge fields.

These solutions have the property that they have 4-branes and 2-branes with non-zero intersection number and yet no zero branes. If we use only $U(1)$ fields, there is necessarily a non-vanishing 0-brane charge related to the intersection number of the 2-branes. It is therefore important to turn on the $SU(N)$ fields to get configurations with these kinds of charges.

The following is a construction of solutions with these charges. We consider diagonal fields which break $U(N)$ to $U(k) \times U(l)$. Each block has vanishing $SU(l)$ or $SU(k)$ gauge potentials and a solution may be obtained by taking two copies of the solutions of the type obtained in the previous section. The $SU(N)$ field strengths however are no longer

vanshing. There are now two sets of twists $n_{\mu\nu}^{(l)}$ and $n_{\mu\nu}^{(k)}$. The total twist in the $U(1)$ sector, which corresponds to 2-brane charge is given by

$$n_{\mu\nu} = n_{\mu\nu}^{(l)} + n_{\mu\nu}^{(k)}. \quad (3.7)$$

As before we will take all twists except those in the (12) and (34) planes to vanish. The 0-brane charge (instanton number) is then given by

$$P = P^{(l)} + P^{(k)} = \frac{n_{12}^{(l)} n_{34}^{(l)}}{l} + \frac{n_{12}^{(k)} n_{34}^{(k)}}{k}. \quad (3.8)$$

To get the (422) system, we take an ansatz for the non-zero $U(N)$ fields of the form,

$$\begin{aligned} F_{12} &= \text{Diag}(B_{12}, 0) \\ F_{34} &= \text{Diag}(0, B_{34}). \end{aligned} \quad (3.9)$$

This corresponds to taking $n_{12}^{(k)} = n_{34}^{(l)} = 0$. Flux quantization conditions take the form:

$$\begin{aligned} B_{12} a_1 a_2 &= \frac{2\pi n_{12}^{(l)}}{l} \\ B_{34} a_3 a_4 &= \frac{2\pi n_{34}^{(k)}}{k} \end{aligned} \quad (3.10)$$

All other twists $n_{\mu\nu}^{(k)}$ and $n_{\mu\nu}^{(l)}$ are zero. Requiring the traceless parts $F_{ij} - \frac{1}{N} \text{tr} F_{ij}$, to be self-dual imposes $B_{12} = -B_{34}$. These equations imply a condition on the box sizes

$$\frac{a_1 a_2}{a_3 a_4} = \frac{n_{12}^{(l)} k}{n_{34}^{(k)} l}. \quad (3.11)$$

The trace part does not need to be self-dual for supersymmetry to be preserved as we see from (2.5). To make sure that these are acceptable solutions we need to find integers $s_\mu^{(l)}, t_\mu^{(l)}$ which satisfy the condition

$$n_{\mu\nu}^{(l)} = s_\mu^{(l)} t_\nu^{(l)} - s_\nu^{(l)} t_\mu^{(l)} \mod l, \quad (3.12)$$

for fixed $n_{\mu\nu}^{(l)}$ and l . This is easily solved in this example, for instance take $s_1^{(l)} = n_{12}^{(l)}$ and $t_2^{(l)} = 1$ with other s, t 's set to zero.

The instanton number coming from the $SU(N)$ part of the theory is $-\frac{n_{12}^{(l)} n_{34}^{(k)}}{N}$ and the contribution coming from the $U(1)$ part is the opposite. These two contributions can

in general be fractional. If we choose the twists $n_{(12)}^{(l)} = n_{(34)}^{(k)}$, the box sizes are directly related to the pattern of gauge symmetry breaking.

This configuration then has the charges of 4-branes aligned along directions 1, 2, 3, 4 and 2-branes along 1, 2 and 2-branes aligned along 34. T-duality along (12) direction can be performed giving a system with (420) charges, which will be discussed further below. T-duality along 1 and 3 gives a system with charges of two-branes along (24), (14) and (23). This may be interpreted as a system obtained by two two-branes whose projections along (24) are k and l respectively. One of them is rotated off the (24) plane by rotations in the (12). The other system is rotated off the (24) plane by rotations in the (34) plane.

3.3. (4220) solutions with non-trivial $SU(N)$ gauge fields.

These bound states of 4, 2, and 0-branes have the property that the element of $H^2(T^4, Z)$ defined by the 2-brane charge has non-zero intersection number, but the total intersection number does not have to be zero (unlike the case of section 3.1). Given the discussion in the last section, it is clear that there are (4220) solutions with non-vanishing $SU(N)$ fields obtained by breaking $U(N)$ to $U(l) \times U(k)$. The fields live along the diagonals of the $U(l)$ and the $U(k)$.

3.4. (420) solutions

These bound states have the property that the 2-brane charge as an element of the $H^2(T^4, Z)$ has zero intersection number. The (422) systems are T-dual to (420). The existence of the (420) states suggests that we look for solutions which are BPS and which have instanton number in spite of having a twist in only one plane. This would be impossible for a $U(1)$ theory but is possible for $U(N)$ because of the non-trivial interplay between the $U(1)$ and the $SU(N)$ parts. We need a twist in the (12) plane only, say. The key fact which makes this possible is that $SU(N)$ solutions can be constructed which have non-trivial twists in one plane only.

4. Geometrical constraints

The existence of the above solutions requires satisfying some striking constraints. For example for one class of solutions of (422) type, the ratios of box sizes are rational, and the rational number in question is related to the structure of symmetry breaking caused

by the solution. In this section we understand these constraints by relating them to the geometry and supersymmetry of systems of two-branes.

All the above systems could be mapped by appropriate T-dualities to systems carrying 2-brane charge only. This suggests an interpretation in terms of 2 branes possibly at angles. The first type of solution we considered had no symmetry breaking and all the field strengths are constant in spacetime and live in the $U(1)$. They can be interpreted as T-duals of a single brane at an angle to the directions where T duality is performed. The other solutions, which involve non-trivial $SU(N)$ gauge fields and which break the $U(N)$ to $U(k) \times U(l)$ can be obtained by dualizing a configuration of 2 sets of two branes at a relative angle. since we have two (sets of) 2-branes treated differently the pattern of symmetry breaking is easily understood. The linear form of the gauge potentials as a function of the coordinates, is interpreted after T-duality in terms of rotations [17]. The fact that the contribution to the instanton number coming from each block is determined in terms of the fluxes corresponds to the condition which says that the two-brane configuration is not self-intersecting. The integrality of the net instanton number is also evidence that this picture is correct. If we had fractional zero brane charge, in the T-dual system this would mean that there was fractional two brane charge in the system of two brane oriented at an angle, which would be a contradiction ¹.

Having outlined the general arguments in favour of the relation between these solutions and systems of two-branes, we describe in some more detail how the constraints match in two classes of examples.

4.1. the (4220) with $U(1)$ gauge fields only

This system has the charges of 4-branes wrapped on (1234), two-branes wrapped on (12), two-branes on (34), and 0-branes. A T-duality along (12) gives another (4220) system. A T-duality along (13) gives a system with charges of two-branes along (24), (14), (23) and (13) planes. This can be interpreted in terms of a system of N two-branes which start from the configuration parallel to the (24) plane and get rotated by angles θ_{12} and θ_{34} which mix respectively the directions (12) and (34). Since there is only one set of branes, we do not expect, from the worldsheet description, any constraints from supersymmetry. The only constraints come from the requirement that the 2-brane charge is an integral combination of the 2-cycles of the torus. These correspond to the flux quantization conditions in the

¹ This point was developed in a discussion with S. Mathur.

Yang Mills side. The absence of any extra constraints coming from the susy corresponds to the fact that the field strength does not have to be self dual to guarantee BPS saturation.

Given a two brane at an angle, it is clear that we can increase all the charges by a factor S by just having a stack of S two-branes parallel to each other. In Yang Mills this requires that given any solution with fluxes, instanton number and s_μ , t_μ etc. specified, we should be able to scale up N , the fluxes and the instanton number by the factor S , and recover a solution with the new parameters. This is indeed possible, if we accompany it by a scaling of s_μ by a factor of S , leaving the t fixed.

A map of the two brane to the target space is given by

$$X^\mu(p, q) = b^\mu m_p^\mu p + b^\mu m_q^\mu q, \quad (4.1)$$

where p and q are defined on the interval $[0, 1]$, and b_μ are the dimensions of the torus related by T duality along 1 and 3 to the torus containing the (4220) system. The 2-brane charges $Q_{\mu\nu}$ are equal to the covering numbers in the $\mu\nu$ plane:

$$Q_{\mu\nu} = (m_p^\mu m_q^\nu - m_q^\mu m_p^\nu). \quad (4.2)$$

Because these charges are related to covering numbers through (4.2), they satisfy the constraint

$$\epsilon_{\mu\nu\alpha\beta} Q_{\mu\nu} Q_{\alpha\beta} = 0. \quad (4.3)$$

This is just the condition that the two-cycle defined by the surface of the two-brane is not self intersecting. After undoing the T duality in the 1 and 3 directions, this corresponds to constraints in Yang Mills theory relating the instanton number to the fluxes and the rank of the gauge group. For instance in the case in which the $SU(N)$ field strength vanishes, and there are non-vanishing $U(1)$ fluxes n_{12} and n_{34} , the instanton number is given by $P = n_{12}n_{34}/N$. After T duality in the 1 and 3 directions this corresponds to $Q_{13} = Q_{14}Q_{23}/Q_{24}$, which is equivalent to the constraint (4.3).

4.2. the (422) system

The geometrical system which is T-dual (by a T duality along the directions 1 and 3) to the class of solutions with (422) charges which we described in section 3, consists of a brane $B^{(l)}$ (corresponding to $U(l)$) which lies along the 2-cycle $Q_{24}^{(l)}(24) + Q_{14}^{(l)}(14)$ and a brane $B^{(k)}$ (corresponding to $U(k)$) which lies along the 2-cycle $Q_{24}^{(k)}(24) + Q_{23}^{(k)}(14)$. The geometric constraint (4.3) is separately satisfied in each unbroken subgroup. This can be

seen from (3.8). The 2-brane $B^{(l)}$ corresponding to $U(l)$ subgroup has the nonvanishing charges

$$\begin{aligned} Q_{14}^{(l)} &= n_{12}^{(l)}, \\ Q_{24}^{(l)} &= l. \end{aligned} \tag{4.4}$$

The 2-brane $B^{(k)}$ corresponding to the $U(k)$ subgroup has non-vanishing charges :

$$\begin{aligned} Q_{23}^{(k)} &= n_{34}^{(k)}, \\ Q_{24}^{(k)} &= k. \end{aligned} \tag{4.5}$$

The brane $B^{(l)}$ has been rotated off the (24) plane by an angle θ_{12} in the (12) plane

$$\tan\theta_{12} = \frac{Q_{14}^{(l)}b_1}{Q_{24}^{(l)}b_2} \tag{4.6}$$

The brane $B^{(k)}$ has been rotated off the (24) plane by an angle θ_{34} in the (34) plane

$$\tan\theta_{34} = \frac{Q_{23}^{(l)}b_3}{Q_{24}^{(k)}b_4} \tag{4.7}$$

Using the T-duality relations (in units where $4\pi(\alpha')^2 = 1$):

$$\begin{aligned} b_1 &= \frac{1}{a_1}, b_2 = a_2, b_3 = \frac{1}{a_3}, b_4 = a_4 \\ 2\pi \tan\theta_{12} &= B_{12}, 2\pi \tan\theta_{34} = B_{34}, \end{aligned} \tag{4.8}$$

we see that the geometrical equations (4.6) and (4.7) are precisely the flux quantization conditions in (3.10).

It follows by considerations starting from the worldsheet formulation of 2-branes [13] that such a system with $\theta_{12} = \theta_{34}$ is supersymmetric. Therefore the pair of branes can be supersymmetric and compatible with the periodicities defining the torus, if the box sizes satisfy the constraint which is precisely the one in (3.11).

5. Mass formula and quantum ground states of $SU(N)$ Yang Mills

The supersymmetry algebra can be used to obtain the masses of the bound states with charges (4220). It has been obtained in [6] by considering the exchange of gravitons

between the bound states. We can also obtain it by relating it to a T-dual system of minimal area 2-branes. The rotated 2 brane has an area which is given by :

$$A^2 = ((Q_{24}b_2b_4)^2 + (Q_{14}b_1b_4)^2 + (Q_{23}b_2b_3)^2 + (Q_{13}b_1b_3)^2) \quad (5.1)$$

In our units ($4\pi(\alpha')^2 = 1$) all the brane tensions are equal, and gauge fields under T-duality are 2π times the corresponding coordinates. The mass of the two-brane can then be expressed in terms of the above data and the string coupling. Under this T duality, Q_{24} becomes the number of 4-branes, i.e N in the gauge theory, Q_{14} becomes two brane number in the (34) plane or n_{12} in the gauge theory language, Q_{23} becomes 2-brane charge in the (12) plane or the flux number n_{34} in the gauge theory language; and finally Q_{13} becomes the zero brane charge or instanton number. The box sizes are related as in (4.8). Recalling the transformation of the string coupling under T duality we can express the mass in terms of the charges of the (4220) system.

$$m^2 = (Q_4^2 + (Q_2^{(12)})^2 + (Q_2^{(34)})^2 + Q_0^2). \quad (5.2)$$

If we consider fluxes such that $n_{12}n_{34}/N$ is of order 1, and we expand in large N , then the above expression simplifies to an expression with three terms which contain the two-brane and 4-brane charges quadratically. The term coming from the instanton number only affects higher orders in the $1/N$ expansion. The contribution from the classical Yang Mills action for the $U(1)$ part correctly gives the excess energy of the two-branes. This means that the contribution from the $SU(N)$ sector is zero. This gives a non-trivial prediction for the energies of states of minimal energy in supersymmetric (with 16 supercharges) $SU(N)$ Yang Mills theory on $T^4 \times R$ with fixed box sizes, and fluxes $\frac{n_{12}n_{34}}{N}$ of order 1, as N goes to infinity. This is the same kind of argument that was used in [2] and [15] to deduce properties of supersymmetric Yang Mills theories. In cases considered in [15], where only an electric flux in some definite direction in the torus was present, the questions about ground states of Yang Mills on tori could be reduced to lower dimensional questions because fluctuations in directions transverse to the direction of the flux could be ignored. The questions we are considering here cannot be reduced to questions of quantum mechanics or of $1+1$ dimensional field theory, so they probe more “higher dimensional” properties of the 5-dimensional Yang Mills theory (regulated by its embedding in the theory of 4-branes). A better understanding of these properties is necessary in the context of the matrix model approach of [8] and the approach to compactification explored in [8][18][19][9].

6. Comments and conclusions.

In the above we have discussed in detail supersymmetric solutions where $U(N)$ is left unbroken, and the case where it is broken to $U(k) \times U(l)$. The former were related to systems with one (stack of) non self-intersecting branes aligned along some 2-cycle in T^4 . The latter were related to two (stacks of) branes at a relative angle. The solutions may be generalized to situations where the unbroken symmetry group has more than two factors. Correspondingly on the two-brane side, there are solutions with more than two sets of branes which preserve supersymmetry [13].

T duality on all 4 directions of the T^4 gives relations between instanton moduli spaces of $U(N)$ of instanton number k and instanton moduli spaces of $U(k)$ of instanton number N (together with a reshuffling of two brane fluxes), as mentioned in the physics literature in [4] and [20]. In most of the discussion relating two-branes to the systems of 4-branes we T-dualized in a definite pair of directions, namely the directions 1 and 3. As a consequence the two-brane charges in the (24) plane determined the ranks of the gauge groups involved. If we started from the same two-brane system and T-dualized in the directions 2 and 4, the ranks of the gauge groups would be related to two brane charges in the (13) plane. This suggests that the symmetries which exchange instanton number and rank of gauge group can be made manifest by considering systems of 2-branes that they are dual to.

One of our motivations for putting the torons of [3] in the context of D-brane bound states was to understand the fractional tension strings appearing in the effective string models of black holes [21][22][23][24][25]. The systems we have considered in this paper are simpler and have more supersymmetry than those of interest in these papers, but some qualitative similarities between the behaviour of fractional instantons in the systems we discussed and those of fractional strings entering black hole models may be noticed. In both cases the fractional objects are useful building blocks but enter the physical systems in combinations which have integral net charge. In the black hole context only the integrally charged objects can interact with closed strings. In the systems we studied imposing the full $U(N)$ gauge symmetry required that the net instanton charge be integral. Similar behaviour can be seen in another simple system with fractional branes studied in [26] and [27].

Another comment may be made concerning possible relations to black holes. As emphasized in [28], the effective string models only use the tree level string theory or the conformal field theory. We have seen in section 5 that the systems we studied have simple

behaviour at large N , the classical $U(1)$ energy agreeing with the correct BPS formula in appropriate flux sectors. This suggests that the surprising effectiveness of the tree level effective string model may be related to the “classical” behaviour of large N gauge theories.

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